

## Rainfall-runoff model with non-linear reservoir

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On website <https://www.waterlog.info>

Theory of the RainOff model to be found on <https://www.waterlog.info/rainoff.htm>

### Introduction

The runoff from watershed (hydrologic catchment areas) resulting from rainfall can be simulated using the principles of the non-linear reservoir. The principles can also be applied to the discharge of agricultural drainage systems with pipe drains or ditches in response to recharge by rainfall or irrigation.

Hereunder, the basics of the linear reservoir are dealt with first, where after the non-linear reservoir is introduced. Finally, the use of the reservoir model in agricultural land drainage with pipes or ditches is illustrated.

### The linear reservoir

The hydrology of a linear reservoir (figure 1) is governed by two equations.

1 - flow equation:

$$Q = A.S \quad (1)$$

with  $Q$  in units  $[L/T]$ , where  $L$  is length (e.g. mm) and  $T$  is time (e.g. hour, day)

2 - continuity or water balance equation:

$$R = Q + dS/dT, \text{ with units } [L/T] \quad (2)$$

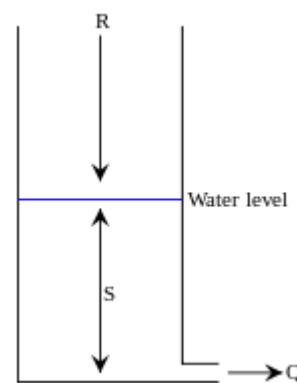


Fig. 1 - A linear reservoir

Definition of symbols used:

$Q$  is the *runoff* or *discharge*

$R$  is the *effective rainfall* or *rainfall excess* or *recharge*

$A$  is the constant *reaction factor* or *response factor* with unit  $[1/T]$

$S$  is the water storage with unit  $[L]$

$dS$  is a differential or small increment of  $S$

$dT$  is a differential or small increment of  $T$

### Runoff equation

A combination of the two previous equations results in a differential equation whose solution is:

$$Q_2 = Q_1 \exp \{ -A (T_2 - T_1) \} + R [ 1 - \exp \{ -A (T_2 - T_1) \} ] \quad (3)$$

This is the *runoff equation* or *discharge equation*, where  $Q_1$  and  $Q_2$  are the values of  $Q$  at time  $T_1$  and  $T_2$  respectively while  $T_2 - T_1$  (representing  $dT$ ) is a small time step during which the recharge can be assumed constant.

### Computing the total hydrograph

Provided the value of  $A$  is known, the *total hydrograph* can be obtained using a successive number of time steps and computing, with the *runoff equation*, the runoff at the end of each time step from the runoff at the end of the previous time step.

### Unit hydrograph

The discharge may also be expressed as:  $Q = -dS/dT$ . Substituting herein the expression of  $Q$  in equation (1) gives the differential equation  $dS/dT = A.S$ , of which the solution is:  $S = \exp(-A.t)$ . Replacing herein  $S$  by  $Q/A$  according to equation (1), it is obtained that:  $Q = A \exp(-A.t)$ . This is called the instantaneous unit hydrograph (IUH) because the  $Q$  herein equals  $Q_2$  of the foregoing runoff equation using  $R = 0$ , and taking  $S$  as *unity* which makes  $Q_1$  equal to  $A$  according to equation (1).

### Determining the response factor A

When the *response factor*  $A$  can be determined from the characteristics of the watershed (catchment area), the reservoir can be used as a *deterministic model* or *analytical model*. Otherwise, the factor  $A$  can be determined from a data record of rainfall and runoff using the method explained below under *non-linear reservoir*. With this method the reservoir can be used as a *black-box* model.

### Conversions

1 mm/day corresponds to  $10 \text{ m}^3/\text{day}$  per ha of the watershed

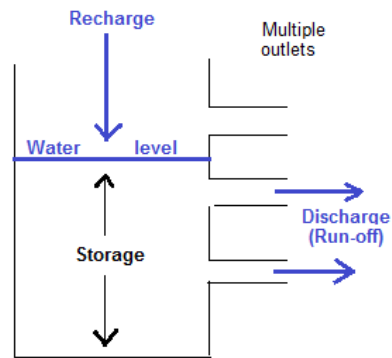
1 l/s per ha corresponds to  $8.64 \text{ mm/day}$  or  $86.4 \text{ m}^3/\text{day}$  per ha

## The non-linear reservoir

Contrary to the linear reservoir, the non linear reservoir has a reaction factor  $A$  that is not a constant, but it is a function of storage  $S$  (figure 2), because the number of outlets increases with increasing  $S$ .

In this figure it can be seen that the discharge increases exponentially when  $S$  increases, while the increase of the water level under influence of a certain recharge reduces at increasing  $S$  values because the reservoir becomes wider at increasing storage.

The response factor here is not constant as for the linear reservoir and it will be called  $A_q$  instead of  $A$



### Non-linear reservoir

As the water level increases the discharge increases more than proportionally

Figure 2. A non-linear reservoir

## No recharge

During periods without rainfall or recharge, i.e. when  $R = 0$ , the runoff equation (3) reduces to

$$Q_2 = Q_1 \exp \{ - A_q (T_2 - T_1) \} \quad (4)$$

or, using a *unit time step* ( $T_2 - T_1 = 1$ ) and solving for  $A_q$ :

$$A_q = - \ln (Q_2/Q_1)$$

Hence, the reaction or response factor  $A_q$  can be determined from runoff or discharge measurements using *unit time steps* during dry spells ( $R=0$ ).

Figure 3 shows the relation between  $A_q$  ( $\alpha$ ) and  $Q$  for a small valley (Rogbom) in Sierra Leone.

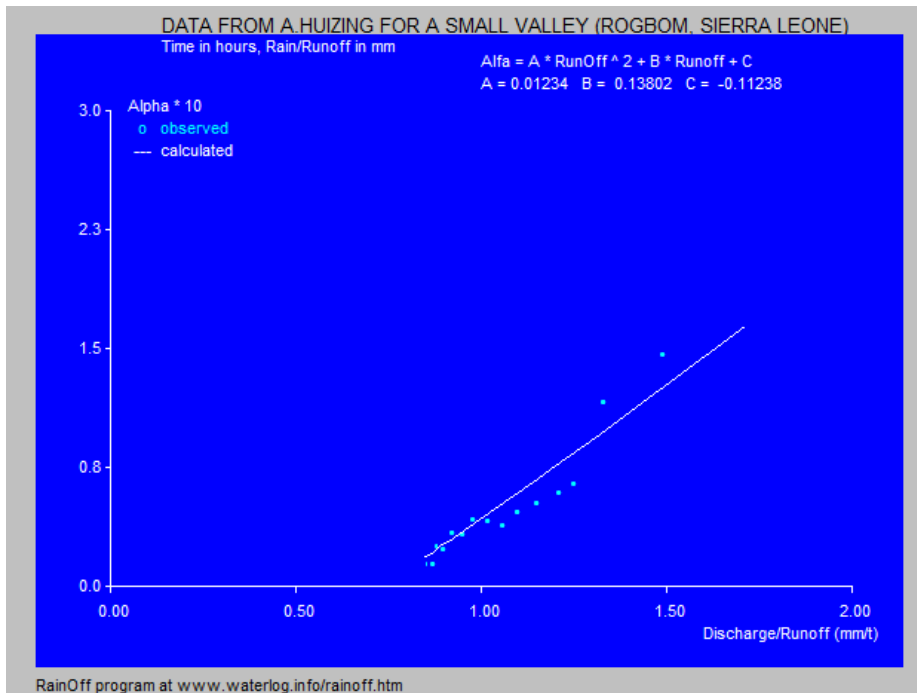


Figure 3. In a non-linear reservoir, the  $Aq$  (alpha) factor changes with increasing runoff ( $Q$ )

## Recharge

The recharge, also called *effective rainfall* or *rainfall excess*, can be modeled by a *pre-reservoir* (figure 4) giving the recharge as *overflow*. The pre-reservoir knows the following elements:

- a maximum storage ( $S_m$ ) with unit length [L]
- an actual storage ( $S_a$ ) with unit [L]
- a relative storage:  $S_r = S_a/S_m$
- a maximum escape rate ( $E_m$ ) with units length/time [L/T]. It corresponds to the maximum rate of *evaporation* plus *percolation* to the groundwater, which will not take part in the runoff process (figure 5, 6)
- an actual escape rate:  $E_a = S_r.E_m$
- a storage deficiency:  $S_d = S_m + E_a - S_a$

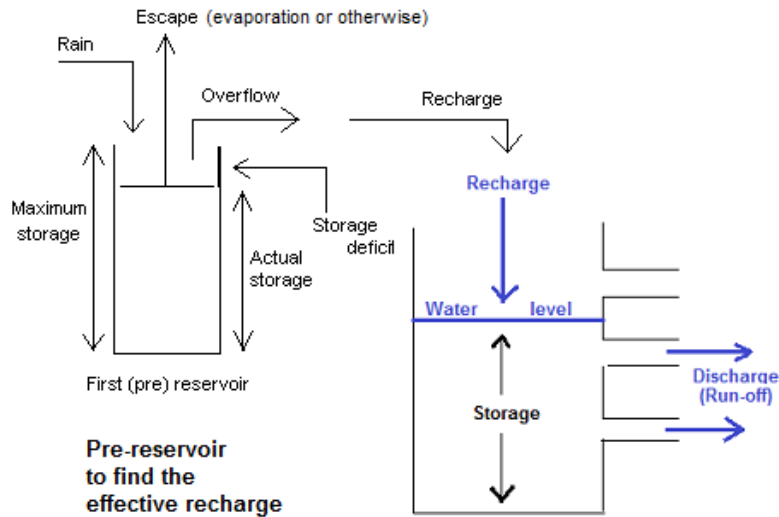


Figure 4. A pre-reservoir to find the effective recharge into the non-linear reservoir.

The recharge  $R$  during a unit time step ( $T_2 - T_1 = 1$ ) can be found from  $R = \text{Rain} - S_d$  (the overflow) when  $\text{Rain} - S_d > 0$ , or  $R = 0$  when  $\text{Rain} - S_d \leq 0$

The actual storage  $S_{a2}$  at the end of a *unit time step* is found as  $S_{a2} = S_{a1} + \text{Rain} - R - E_a$ , where  $S_{a1}$  is the actual storage at the start of the time step.

The Curve Number Method (CN method) gives another way to calculate the recharge. The *initial abstraction* herein compares with  $S_m - S_i$ , where  $S_i$  is the initial value of  $S_a$ .

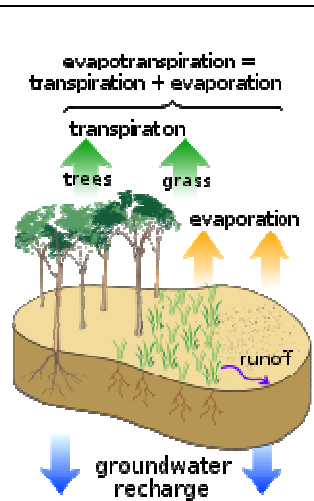


Figure 5. Escape factors evaporation and groundwater recharge.

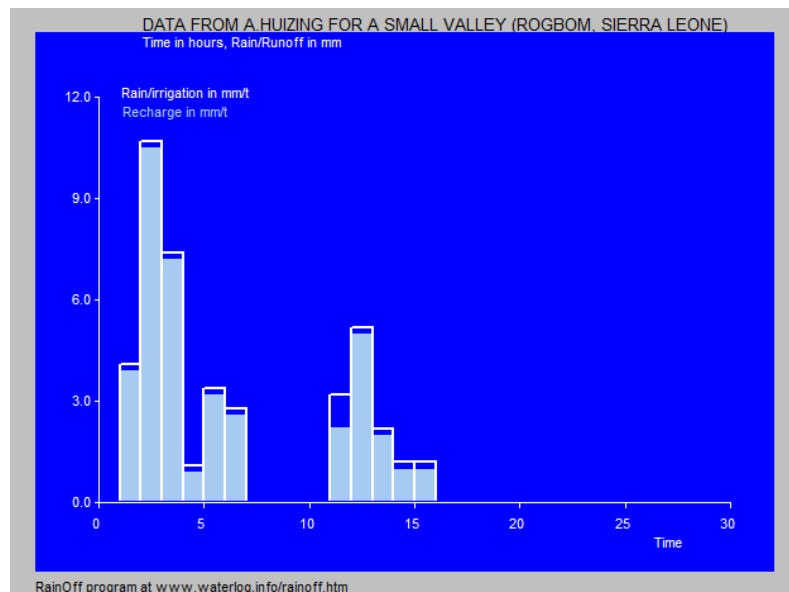


Figure 6. Rainfall converted into recharge for a small valley (Rogbom) in Sierra Leone.

## Complex non-linear reservoir

To increase the versatility of the model, the non-linear reservoir can be made to consist of two parts with different response factors  $A_q$ :  $A_{q1}$  and  $A_{q2}$ . The  $A_{q1}$  function holds for the top part of the reservoir when the higher discharges occur, while the  $A_{q2}$  is valid for the lower (bottom) part. The separation point is called  $Q_s$  (the discharge of the lower part when it is just full)

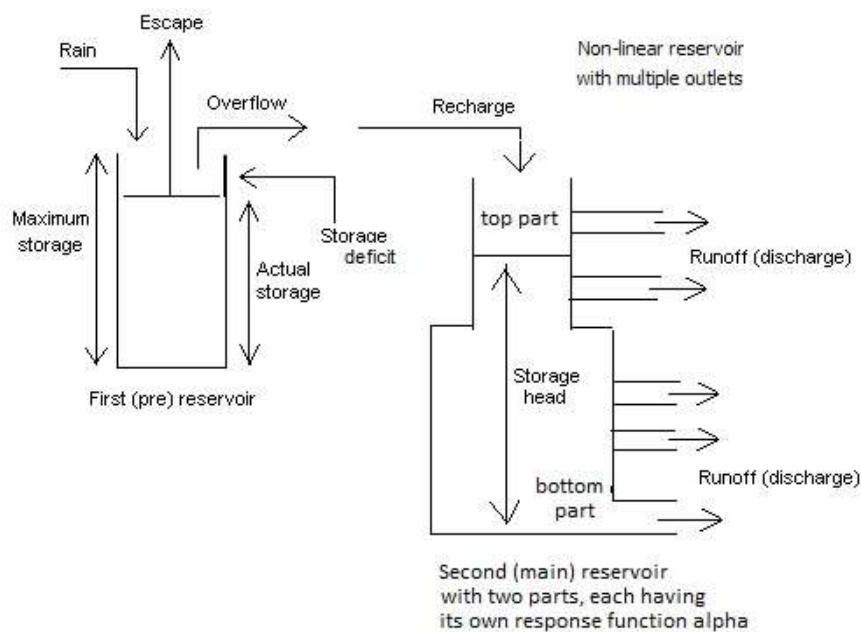


Figure 7. A complex non-linear reservoir consisting of two parts.

The two response functions can consist of two linear equations:

1.  $A_{q1} = A_1 \cdot \text{Runoff} + C_1$  ( $\text{Runoff} > Q_s$ )
2.  $A_{q2} = A_2 \cdot \text{Runoff} + C_2$  ( $\text{Runoff} < Q_s$ )

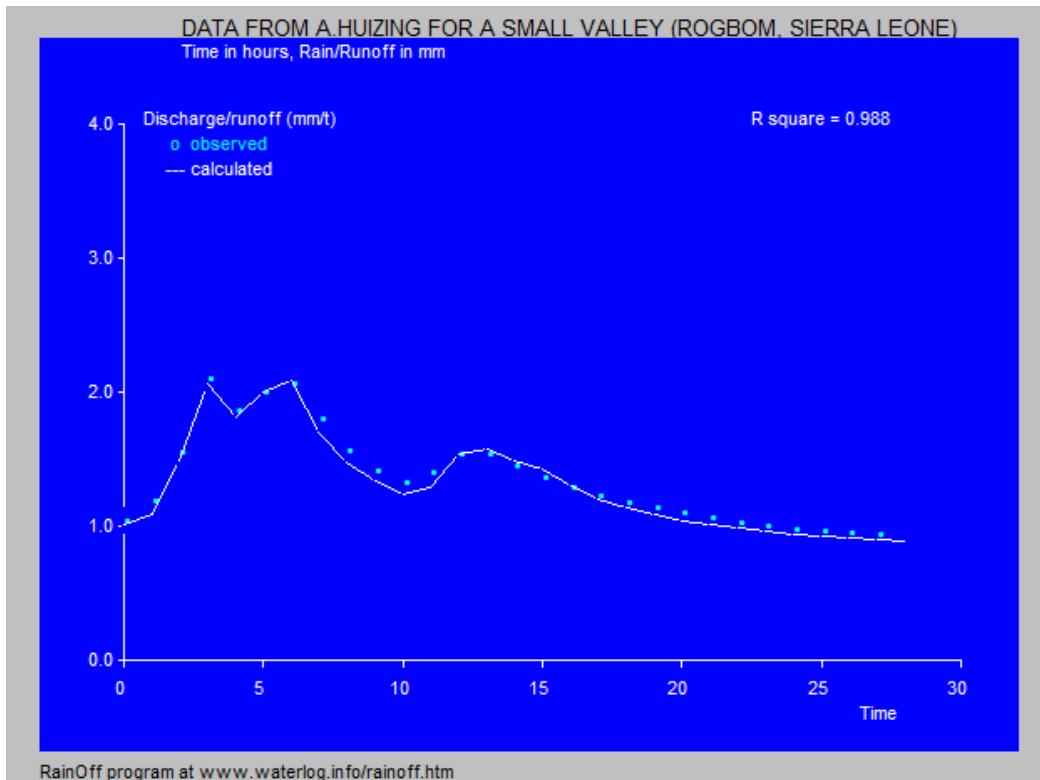


Figure 8. Calibration of observed and calculated (simulated) runoff in a small valley (Rogbom) in Sierra Leone using the RainOff model.

### Agricultural land drainage

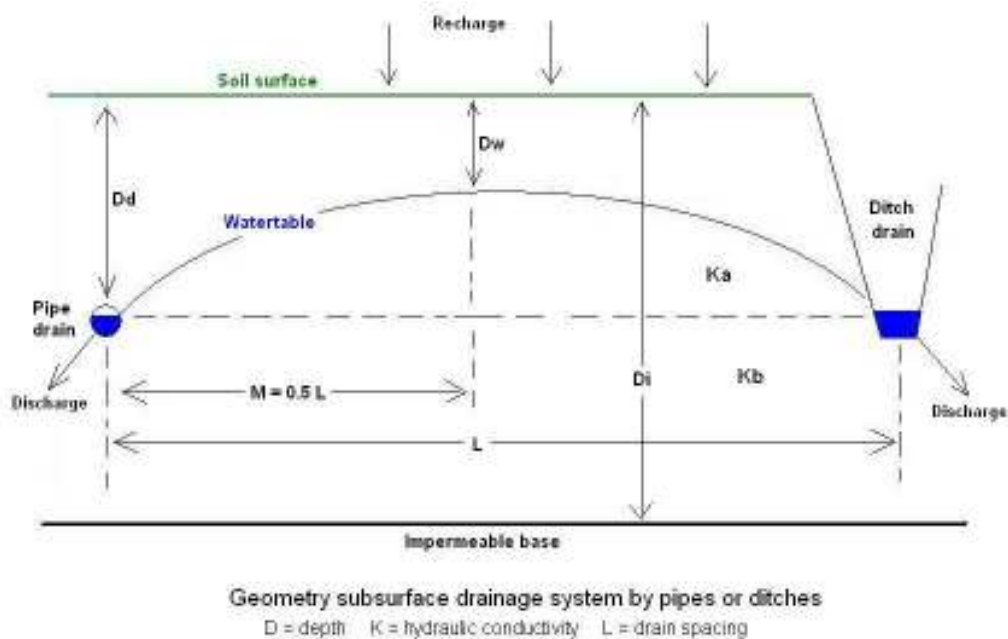


Figure 9.

In the situation of figure 8, the drainage equation of Hooghoudt is applicable.

$$Q = \frac{8K_b \cdot D_e \cdot H}{L^2} + \frac{4K_a \cdot H^2}{L^2}$$

The height (H in m) of the water table midway between the drains above drain level equals  $D_d - D_w$  in figure 9.

$K_a$  and  $K_b$  = hydraulic conductivity above and below drain level respectively (m/day)

$L$  = drain spacing (m)

$D_e$  = equivalent depth of the impermeable layer below drain level. It depends on the actual depth  $D_a = D_i - D_d$  (see figure 8) of the impermeable layer below drain level. The mathematical expression of  $D_e$  in terms of  $D_a$  is shown on the next page.

$Q$  is expressed in m/day.

The drainable storage  $S$  of water midway between the drains equals  $S = P_d \cdot H$  where  $P_d$  is the drainable porosity (in m/m) of the soil, also called effective porosity. In clay soils it normally varies between 2 and 4%, in loamy soils it may vary from 3 to 5% and in sandy loams it may range from 4 to 6% and in sandy soil from 5 to 10%

Replacing in the Hooghoudt equation  $H$  by  $S/P_d$  we get:

$$Q = \frac{8K_b \cdot D_e \cdot S}{P_d \cdot L^2} + \frac{4K_a \cdot S^2}{P_d^2 \cdot L^2}$$

Writing  $Q = A_q \cdot S = \alpha S$  we find

$$\alpha = \frac{8K_b \cdot D_e}{P_d \cdot L^2} + \frac{4K_a \cdot S}{P_d^2 \cdot L^2}$$

or:

$$\alpha = B + A \cdot S$$

where:

$$B = 8K_b \cdot D_e / P_d \cdot L^2$$

$$A = 4K_a / P_d \cdot L^2$$

yielding a reaction (response factor  $\alpha$ ) depending on the storage  $S$  (and therefore also on  $Q$ ), so that we have a non linear reservoir.



## Equivalent depth De

Reference: W.H. van der Molen and J. Wesseling 1991. A solution in closed form and a series solution for the thickness of the equivalent layer in Hooghoudt's drain spacing formula. *Agricultural Water Management* 19, pp. 1-16

$$De = \frac{\pi L/8}{\ln(L/U) + F(x)}$$

where U = wet circumference of the drain (m) and F(x) is a function of

$$x = 2 \pi Da / L$$

When  $x > 1$  then:

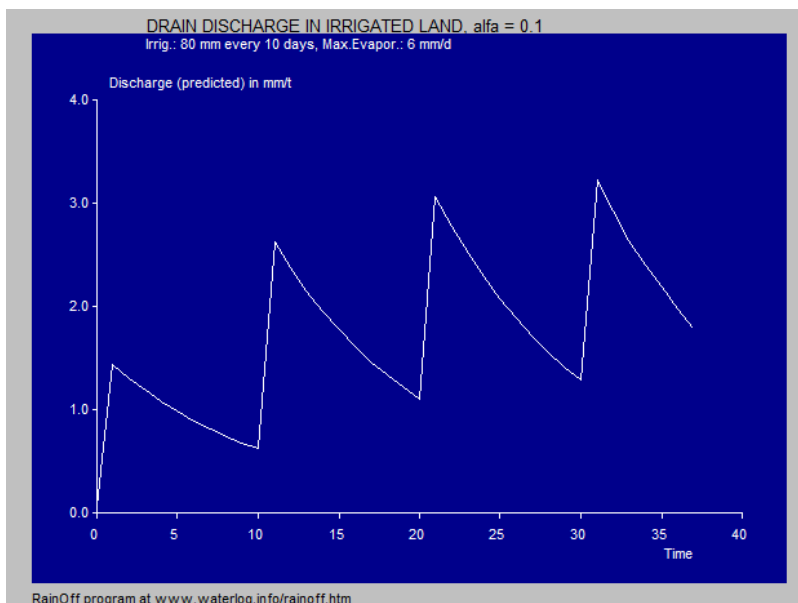
$$F(x) = \frac{4e^{-2x}}{(1 - e^{-2x})} + \frac{4e^{-6x}}{3(1 - e^{-6x})} + \frac{4e^{-10x}}{5(1 - e^{-10x})} + \dots$$

For  $x \leq 1$ :

$$F(x) = \pi^2 / 4x + \ln(x/2\pi)$$

*Note.*

For a half full pipe drain  $U = \pi r$  with  $r$  = drain radius. For a ditch drain U equals bottom width + twice the length of the part of the sides that is under water.



Simulation of drain discharge in arid land with irrigation every 20 days.